

CHAPTER I

Rings and Modules

Introduction. After some preliminaries concerning rings, modules, homomorphisms, direct sums, direct products, and exact sequences, the notions of projective and injective modules are introduced. These notions are fundamental for this book. The basic results here are that each module may be represented as a quotient of a projective module and also as a submodule of an injective one.

In § 4–7 we consider special classes of rings, namely: semi-simple rings, hereditary rings, semi-hereditary rings, and Noetherian rings. It will be seen later (Ch. VII) that for integral domains the hereditary (semi-hereditary) rings are precisely the Dedekind (Prüfer) rings.

1. PRELIMINARIES

Let Λ be a ring with a unit element $1 \neq 0$. We shall consider (left) *modules* over Λ , i.e. abelian groups A with an operation $\lambda a \in A$, for $\lambda \in \Lambda$, $a \in A$ such that

$$\begin{aligned} \lambda(a_1 + a_2) &= \lambda a_1 + \lambda a_2, & (\lambda_1 + \lambda_2)a &= \lambda_1 a + \lambda_2 a, \\ (\lambda_1 \lambda_2)(a) &= \lambda_1(\lambda_2 a), & 1a &= a. \end{aligned}$$

We shall denote by 0 the module containing the zero element alone.

In the special case $\Lambda = Z$ is the ring of rational integers, the modules over Z are simply abelian groups. If Λ is a (commutative) field, they are the vector spaces over Λ .

Given two modules A and B (over the same ring Λ), a *homomorphism* (or linear transformation, or mapping) of A into B is a function f defined on A with values in B , such that $f(x + y) = fx + fy$; $f(\lambda x) = \lambda(fx)$; $x, y \in A$, $\lambda \in \Lambda$. We then write $f: A \rightarrow B$, or $A \rightarrow B$ if there is no ambiguity as to the definition of f . The *kernel* of f is the submodule of A consisting of all $x \in A$ such that $fx = 0$; it will be denoted by $\text{Ker}(f)$ or $\text{Ker}(A \rightarrow B)$. The *image* of f is the submodule of B consisting of all elements of the form fx , $x \in A$; it will be denoted by $\text{Im}(f)$ or $\text{Im}(A \rightarrow B)$.

We also define the *coimage* and *cokernel* of f as follows:

$$\begin{aligned} \text{Coim}(f) &= A/\text{Ker}(f), \\ \text{Coker}(f) &= B/\text{Im}(f). \end{aligned}$$

Of course, f induces an isomorphism $\text{Coim } (f) \approx \text{Im } (f)$ and because of this isomorphism the coimage is very seldom employed.

A homomorphism $f: A \rightarrow B$ is called a *monomorphism* if $\text{Ker } f = 0$; f is called an *epimorphism* if $\text{Coker } f = 0$ or equivalently if $\text{Im } f = B$. If f is both an epimorphism and a monomorphism then f is an *isomorphism* (notation: $f: A \approx B$).

Let A be a module and $\{A_\alpha\}$ a (finite or infinite) family of modules (all over the same ring Λ) with homomorphisms

$$A_\alpha \xrightarrow{i_\alpha} A \xrightarrow{p_\alpha} A_\alpha$$

such that $p_\alpha i_\alpha = \text{identity}$, $p_\beta i_\alpha = 0$ if $\beta \neq \alpha$. We shall say that $\{i_\alpha, p_\alpha\}$ is a *direct family of homomorphisms*.

If we assume that each $x \in A$ can be written as a finite sum $x = \sum i_\alpha x_\alpha$, $x_\alpha \in A_\alpha$, it follows readily that A is isomorphic with the direct sum $\sum A_\alpha$. We therefore say that the family $\{i_\alpha, p_\alpha\}$ yields a *representation of A as a direct sum* of the modules A_α . In this case the mappings $\{p_\alpha\}$ can be defined using the $\{i_\alpha\}$ alone.

If we assume that for each family $\{x_\alpha\}$, $x_\alpha \in A_\alpha$, there is a unique $x \in A$ with $p_\alpha x = x_\alpha$, it follows readily that A is isomorphic with the direct product $\prod A_\alpha$. We therefore say that the family $\{i_\alpha, p_\alpha\}$ yields a *representation of A as a direct product* of the modules A_α . In this case the homomorphisms $\{i_\alpha\}$ can be defined using the $\{p_\alpha\}$ alone.

If the family $\{A_\alpha\}$ is finite, the notions of direct sum and direct product coincide. A finite direct family yields a direct sum (or direct product) representation if and only if $\sum i_\alpha p_\alpha = \text{identity}$.

A sequence of homomorphisms

$$A_m \rightarrow A_{m+1} \rightarrow \cdots \rightarrow A_n, \quad m+1 < n$$

is said to be *exact* if for each $m < q < n$ we have $\text{Im } (A_{q-1} \rightarrow A_q) = \text{Ker } (A_q \rightarrow A_{q+1})$. Thus $A \rightarrow B$ is a monomorphism if and only if $0 \rightarrow A \rightarrow B$ is exact and an epimorphism if and only if $A \rightarrow B \rightarrow 0$ is exact. We shall also allow sequences which extend to infinity to the left or to the right or in both directions.

In particular, we shall consider exact sequences

$$(*) \quad 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0.$$